Computational Complexity of Description Logics:

a Friendly Introduction to Some Interesting Phenomena

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Warm up

Which of the following subsumptions hold?

(1) r some (A and B) is subsumed by r some A

$$\exists r.(A \sqcap B) \sqsubseteq \exists r.A$$

(2) (r some A) and (r only B) is subsumed by r some B

$$\exists r.A \sqcap \forall r.B \sqsubseteq \exists r.B$$

(3) r only (A and not A) is subsumed by r only B

$$\forall r.(A \sqcap \neg A) \sqsubseteq \forall r.B$$

(4) r some (r only A) is subsumed by r some (r some (A or not A))

$$\exists r.(\forall r.A) \sqsubseteq \exists r.(\exists r.(A \sqcup \neg A)$$

(3) r only (A and B) is subsumed by (r only A) and (r only B)

$$\forall r.(A \sqcap B) \sqsubseteq \forall r.A \sqcap \forall r.B$$

(4) r some B is subsumed by r only B

$$\exists r.B \sqsubseteq \forall r.A$$

- we will discuss a lot of things
- but also leave out a lot
- ...please **ask** if you have a question!

Reminder: Standard DL Reasoning Problems

Given an ontology $\mathcal{O} = (\mathcal{T}, \mathcal{A})$,

- is $\mathcal{O}$ consistent? $\mathcal{O} \models \top \sqsubseteq \bot$?
- is $\mathcal{O}$ coherent? is there concept name A with $\mathcal{O} \models A \sqsubseteq \bot$?
- compute concept hierarchy! for all concept names A, B: $\mathcal{O} \models A \sqsubseteq B$?
- classify individuals! for all concept names A, individual names b: $\mathcal{O} \models b \colon B$?

Theorem 1 Let $\mathcal{O}$ be an ontology and a an individual name **not** in $\mathcal{O}$. Then

1. C is satisfiable w.r.t. $\mathcal{O}$ iff $\mathcal{O} \cup \{a \colon C\}$ is consistent
2. $\mathcal{O}$ is coherent iff, for each concept name A, $\mathcal{O} \cup \{a \colon A\}$ is consistent
3. $\mathcal{O} \models A \sqsubseteq B$ iff $\mathcal{O} \cup \{a \colon (A \sqcap \neg B)\}$ is **not** consistent
4. $\mathcal{O} \models b \colon B$ iff $\mathcal{O} \cup \{b \colon \neg B\}$ is **not** consistent

➥ a **decision procedure** for consistency decides **all** standard DL reasoning problems

Decision Procedure

- A **problem** is a set $P \subseteq M$
  - e.g., M is the set of all $\mathcal{ALC}$ ontologies,
  - $P \subseteq M$ is the set of all **consistent** $\mathcal{ALC}$ ontologies
  - ...and **the problem** P is to decide whether, for a given $m \in M$, we have $m \in P$

- An algorithm is a **decision procedure** for a problem $P \subseteq M$ if it is
  - **sound** for P: if it answers "$m \in P$", then $m \in P$
  - **complete** for P: if $m \in P$, then it answers "$m \in P$"
  - **terminating**: it stops after finitely many steps on any input $m \in M$

Why does "sound and complete" not suffice for being a decision procedure?

The tableau algorithm for $\mathcal{ALC}$ ontologies

Earlier: Anni explained a tableau algorithm for $\mathcal{ALC}$

Input: $\mathcal{ALC}$ TBox $\mathcal{T}$, $\mathcal{ALC}$ concept name C

Output: "yes" if C is satisfiable wrt. $\mathcal{T}$
"no" if not

Is this algorithm

- sound?
- complete?
- terminating?
- ...and how long does it run?

Properties of our tableau algorithm

Lemma 1: Let $\mathcal{O} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ ontology in NNF. Then

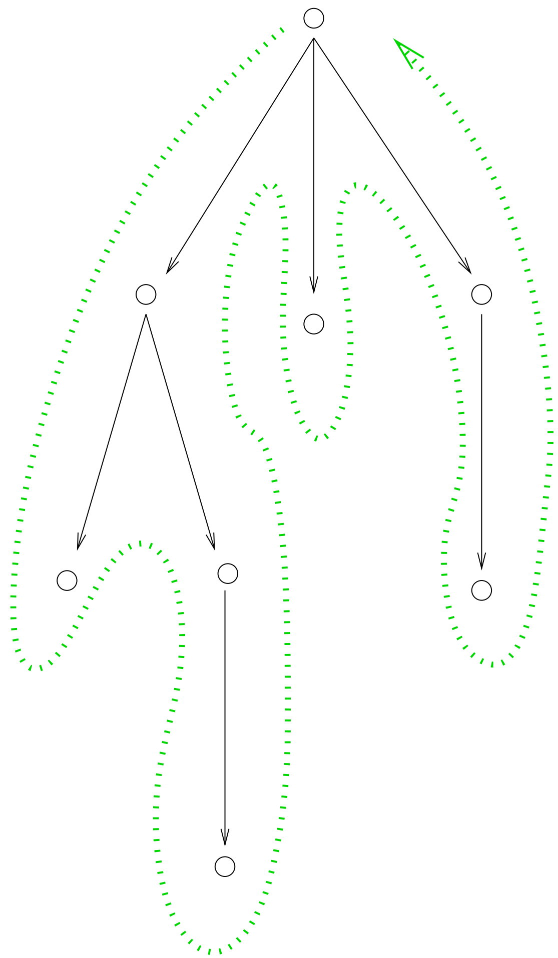
1. the algorithm terminates when applied to $\mathcal{T}$ and C
2. if the rules generate a complete & clash-free ABox, then C is satisfiable wrt. $\mathcal{T}$
3. if C is satisfiable wrt. $\mathcal{T}$, then the rules generate a clash-free & complete ABox

Corollary 1:

1. Our tableau algorithm **decides satisfiability** of $\mathcal{ALC}$ concepts wrt. TBoxes.
2. Satisfiability of $\mathcal{ALC}$ concepts (no TBox!) is decidable in **PSpace**.
3. Satisfiability of $\mathcal{ALC}$ concepts wrt. TBoxes is decidable in **ExpSpace**.
4. $\mathcal{ALC}$ concepts have the **finite model property**
   i.e., every consistent ontology has a **finite** model.
5. $\mathcal{ALC}$ concepts have the **tree model property**
   i.e., every consistent ontology has a **tree** model.

Regarding Corollary 1.2

If we start the algorithm with $\{a : C\}$
to test satisfiability of C, and
construct ABox in non-deterministic depth-first manner
rather than constructing set of ABoxes
so that we only consider a single ABox and
re-use space for branches already visited,
mark $b : \exists R.C \in \mathcal{A}$ with "todo" or "done"



we can run tableau algorithm (even without blocking) in **polynomial space**:

- ABox is of depth bounded by $|C|$, and
- we keep only a single branch in memory at any time.

Regarding Corollary 1.3

If we start the algorithm withwith $\{a : C\}$ and $\mathcal{T}$
to test satisfiability of C wrt. $\mathcal{T}$, and
construct ABox in non-deterministic depth-first manner
rather than constructing set of ABoxes
so that we only consider a single ABox

we can run tableau algorithm in **exponential space**:

- number of individuals in ABox is bounded by $2^{\#\,\mathsf{sub}(\mathcal{T})}$

This is **not** optimal: consistency of $\mathcal{ALC}$ ontologies is decidable
in **exponential time**, in fact **ExpTime**-complete.

A tableau algorithm for $\mathcal{ALC}$: Summary

The tableau algorithm presented here

- ➔ **decides** consistency of $\mathcal{ALC}$ ontologies, and thus also
- ➔ all other standard reasoning problems
- ➔ uses **blocking** to ensure termination, and
- ➔ can be implemented as such or
  using a **non-deterministic** alternative for the $\sqcup$-rule and backtracking.
- ➔ uses **P/Exp-Space**
- ➔ can be implemented in various ways,
  - order/priorities of rules
  - data structure
  - etc.
- ➔ is amenable to optimisations...

Implementing the $\mathcal{ALC}$ Tableau Algorithm

Naive implementation of $\mathcal{ALC}$ tableau algorithm is **doomed to failure:**

It constructs a

- **set of ABoxes**,
- each ABox being of possibly **exponential size**, with possibly exponentially many individuals (see binary counting example)
- in the presence of a GCI such as $\top \sqsubseteq (C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcup D_n)$ and exponentially many individuals, algorithm might generate double exponentially many ABoxes

$\rightsquigarrow$ requires double exponential space or

- use non-deterministic variant and backtracking to consider one ABox at a time

$\rightsquigarrow$ requires exponential space

Implementing the $\mathcal{ALC}$ Tableau Algorithm

Optimisations are crucial

concern every aspect of the algorithm

help in "many" cases (which?)

are implemented in various **DL reasoners**

e.g., FaCT++, Pellet, RacerPro

In the following: a selection of some vital optimisations

Optimising the $\mathcal{ALC}$ Tableau Algorithm: Classification

Reasoners provides service **“classify all concept names in $\mathcal{T}$”**, i.e.,

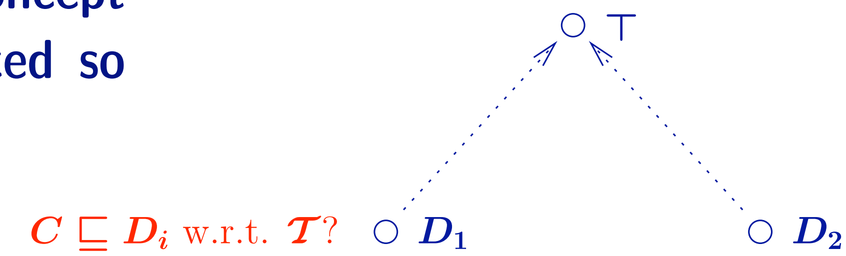
for all concept names C, D in $\mathcal{T}$, reasoner decides **does** $\mathcal{T} \models C \sqsubseteq D$?

$\rightsquigarrow$ test consistency of $\mathcal{T} \cup \{a\colon (C \sqcap \neg D)\}$

$\rightsquigarrow$ n^2 consistency tests!

Goal: reduce number of consistency tests when classifying TBox

Idea 1: “trickle” new concept C into hierarchy computed so far



Optimising the $\mathcal{ALC}$ Tableau Algorithm: Classification II

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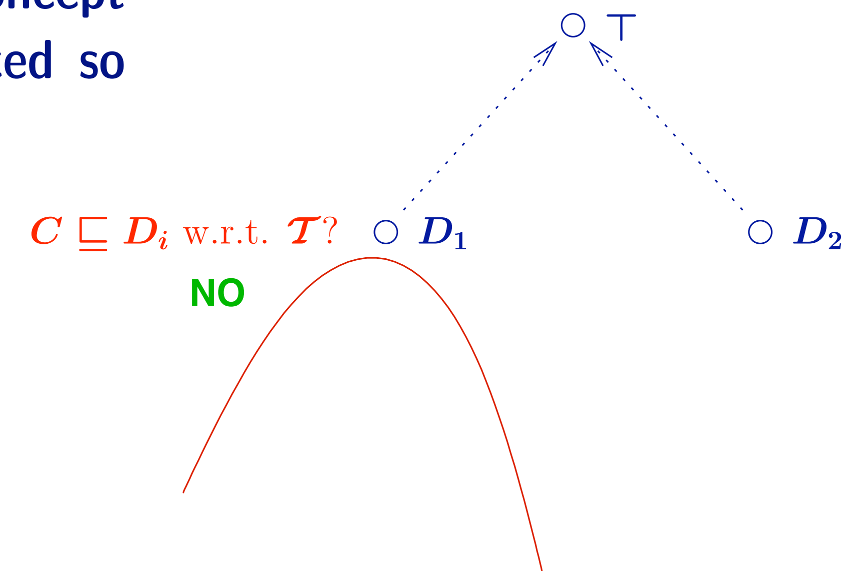

Optimising the $\mathcal{ALC}$ Tableau Algorithm: Classification III

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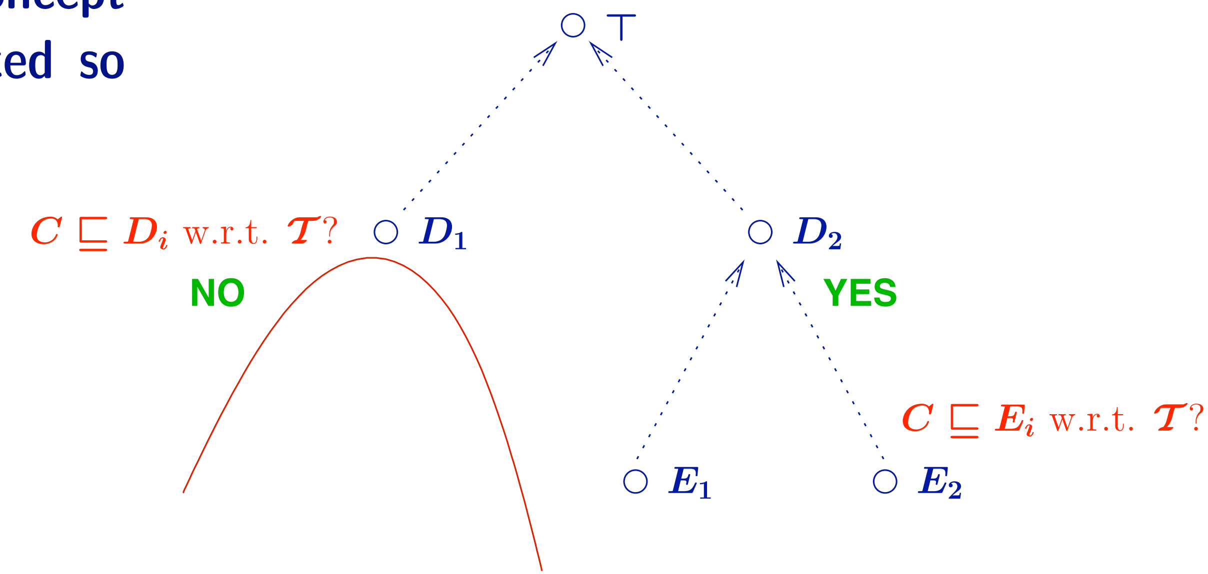

Optimising the $\mathcal{ALC}$ Tableau Algorithm: Classification IV

Reasoners provides service **“classify all concept names $\mathcal{T}$”**, i.e.,

for all concept names C, D in $\mathcal{T}$, reasoner decides **does** $\mathcal{T} \models C \sqsubseteq D$?

$\rightsquigarrow$ test consistency of $\mathcal{T} \cup \{a\colon (C \sqcap \neg D)\}$

$\rightsquigarrow$ n^2 consistency tests!

Goal: reduce number of consistency tests when classifying TBox

Idea 2:

- maintain graph with a node for each concept name
- edges representing subsumption, disjointness ($\mathcal{T} \models A \sqsubseteq \neg B$), and non-subsumption
- initialise graph with all “obvious” information in $\mathcal{T}$
- to avoid testing subsumption, exploit
  - all info in ABox during tableau algorithm to update graph
  - transitivity of subsumption and its interaction with disjointness

Optimising the $\mathcal{ALC}$ Tableau Algorithm: Absorption

Remember: for $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$,
each individual x will have n disjunctions $x\colon (\dot{\neg} C_i \sqcup D_i)$ due to

GCI-rule: if $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$
replace $\mathcal{A}$ with $\mathcal{A} \cup \{a\colon (\dot{\neg} C_1 \sqcup D_1) \sqcap (\dot{\neg} C_2 \sqcup D_2) \sqcap \ldots \sqcap (\dot{\neg} C_n \sqcup D_n)\}$

Problem: high degree of **choice** and huge **search space**
blows up set of ABoxes...we can do better:

2GCI-rule: if $C \sqsubseteq D \in \mathcal{T}$, a is not blocked, and
if C is a concept name, $a\colon C \in \mathcal{A}$ but $a\colon D \notin \mathcal{A}$,
replace $\mathcal{A}$ with $\mathcal{A} \cup \{a\colon D\}$
else if $a\colon (\dot{\neg} C \sqcup D) \notin \mathcal{A}$ for a in $\mathcal{A}$,
replace $\mathcal{A}$ with $\mathcal{A} \cup \{a\colon (\dot{\neg} C \sqcup D)\}$

Problem: still possibly high degree of **choice** and huge **search space**...

Optimising the $\mathcal{ALC}$ Tableau Algorithm: Absorption

Observation: many GCIs are of the form $A \sqcap \ldots \sqsubseteq C$ for concept name A

e.g., $\mathtt{Human} \sqcap \ldots \sqsubseteq C$ or $\mathtt{Device} \sqcap \ldots \sqsubseteq C$

Idea: **localise** GCIs to concept **names** by transforming

$A \sqcap X \sqsubseteq C$ into equivalent $A \sqsubseteq \neg X \sqcup C$

e.g., $\mathtt{Human} \sqcap \exists \mathtt{owns.Pet} \sqsubseteq C$ becomes $\mathtt{Human} \sqsubseteq \neg \exists \mathtt{owns.Pet} \sqcup C$

For "absorbed" $\mathcal{T} = \{A_i \sqsubseteq D_i \mid 1 \leq i \leq n_1\} \cup \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n_2\}$
the second, non-deterministic choice in GCI-rule is taken only n_2 times.

2GCI-rule: if $C \sqsubseteq D \in \mathcal{T}$, a is not blocked, and
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Observations: If no GCI is absorbable, nothing changes

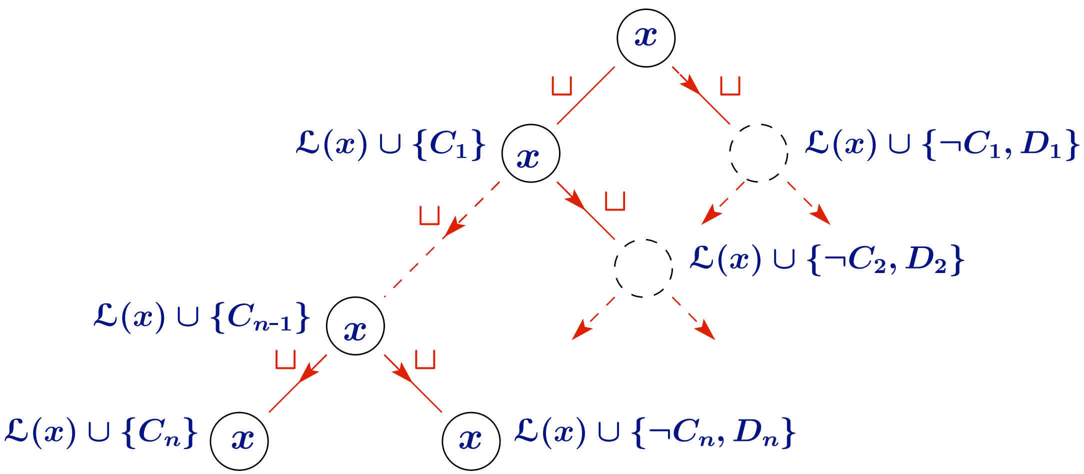
Each absorption saves 1 disjunction **per individual outside** A_i,

in the best case, this avoids **almost all** disjunctions from TBox axioms!

Remember If a **clash** is encountered, **non-deterministic** algorithm **backtracks**

i.e., returns to last non-deterministic choice and
tries other possibility

Example $x: \exists R.(A \sqcap B) \sqcap ((C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcup D_n)) \sqcap \forall R.\neg A$



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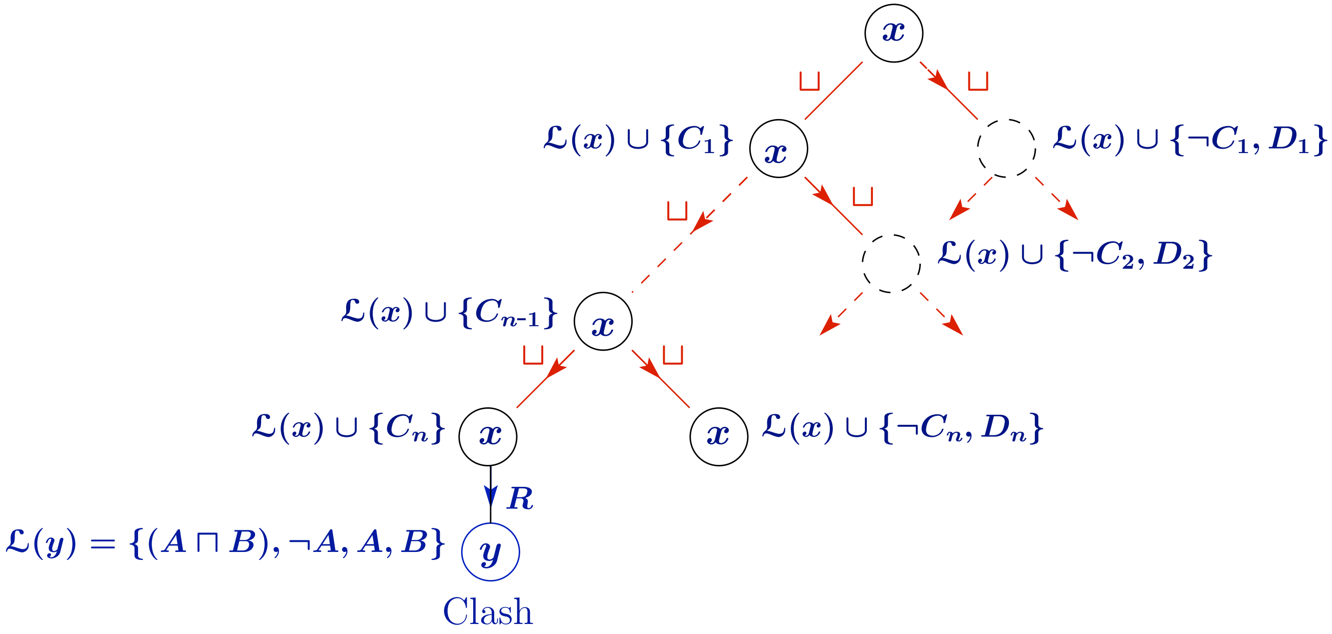

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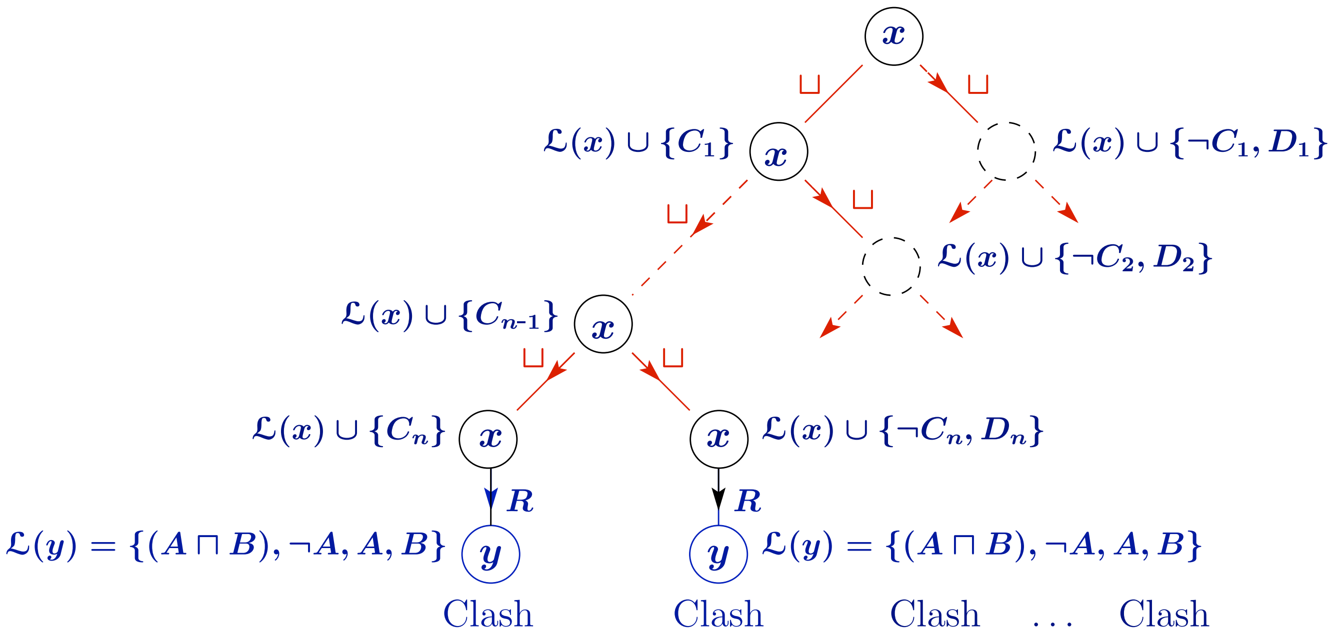

Optimising the $\mathcal{ALC}$ Tableau Algorithm: Backjumping

Remember If a **clash** is encountered, **non-deterministic** algorithm **backtracks**

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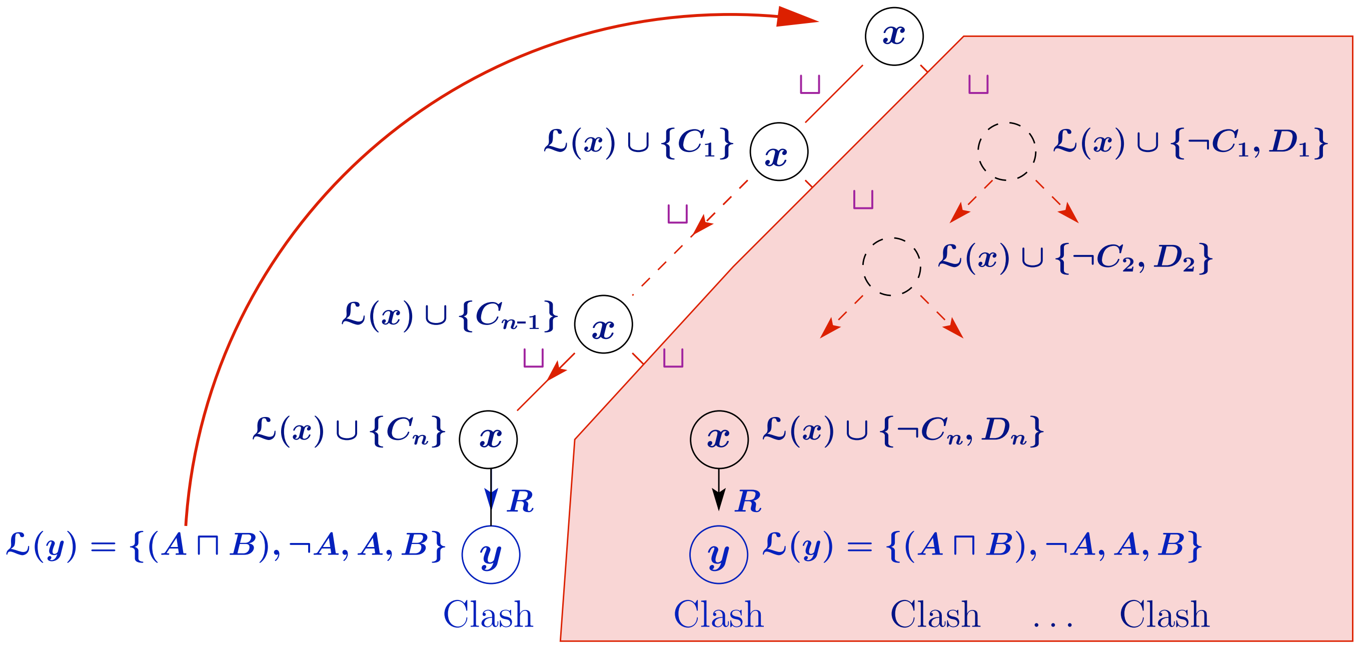

Optimising the $\mathcal{ALC}$ Tableau Algorithm: SAT Optimisations

Finally: $\mathcal{ALC}$ extends **propositional logic**

$\rightsquigarrow$ heuristics developed for **SAT** are relevant

Summing up: optimisations are possible at each aspect of tableau algorithm

can dramatically enhance performance

$\rightsquigarrow$ do they interact?

$\rightsquigarrow$ how?

$\rightsquigarrow$ which combination works best for which "cases"?

$\rightsquigarrow$ is the optimised algorithm still correct?

... check out ORE 2013 results & our "Robustness" paper at DL 2013

...now for some proper computational complexity...

Computational Complexity

We have seen 1 algorithm that runs

- in **PSpace** without a TBox
- in non-deterministic **ExpSpace** with a TBox

...can we do better? How can we tell? ...perhaps try much harder, think much longer?

...how do we show that our algorithm is **optimal**? And what does that mean anyway?

$\rightsquigarrow$ look at **complexity**...

Complexity

We distinguish between

- **cognitive complexity:**
  - e.g., how hard is it, for a human, to determine/understand $\mathcal{O} \models^? C \sqsubseteq D$
  - interesting, little understood topic
  - relevant to provide tool support for ontology engineers
- **computational complexity:**
  - e.g., how much time/space do we need to determine $\mathcal{O} \models^? C \sqsubseteq D$
  - well understood topic
  - loads of results thanks to relationships DL - FOL - Modal Logic
  - relevant to understand
    - trade-off between expressivity (of a DL) and complexity of reasoning
    - whether a given algorithm is optimal/can be improved

Computational Complexity: Decision Problems

Decision problem:

- is a subset $P \subseteq M$
- e.g., $P =$ the set of consistent $\mathcal{ALC}$ ontologies and
  $M =$ the set of all $\mathcal{ALC}$ ontologies
- think of it as **black box** with
  - input $m \in M$
  - output "*yes*" if $m \in P$
    "*no*" if $m \notin P$

(Polynomial) reduction from $P \subseteq M$ to $P' \subseteq M'$ is a (polynomial) function π:

- $\pi : M \longrightarrow M'$
- $m \in P$ iff $\pi(m) \in P'$
- e.g., our translation $t()$ from $\mathcal{ALC}$ to FOL
- e.g., our reduction from subsumption to ontology consistency

Decision problem:
- is a subset $P \subseteq M$
- think of it as **black box** with
  - input $m \in M$
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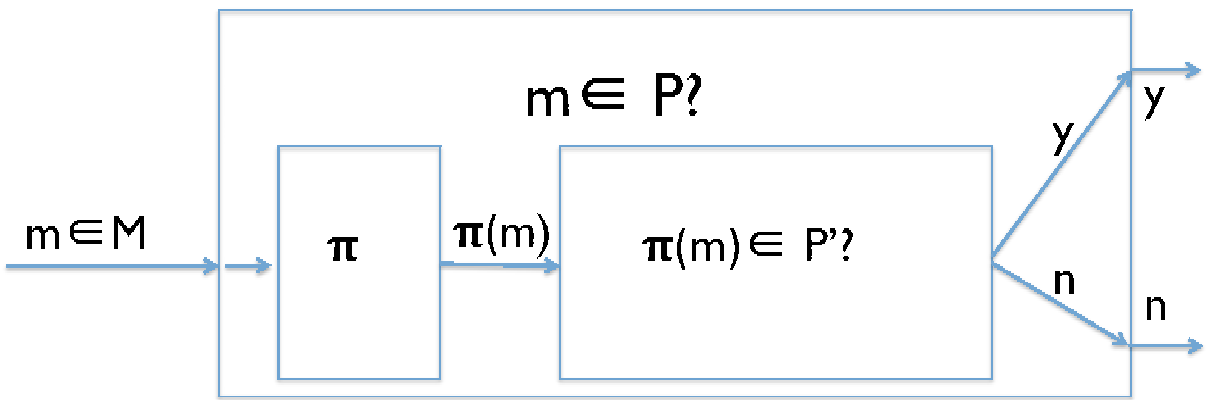

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- is a subset $P \subseteq M$
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(Polynomial) reduction from $P \subseteq M$ to $P' \subseteq M'$ is a (polynomial) function π:
- $\pi : M \longrightarrow M'$ with $m \in P$ iff $\pi(m) \in P'$

Fact: if $P \subseteq M$ is reducible to $P' \subseteq M'$, then
P is at most as hard/complex[a] as P'
because P can be solved by solving P' via π

[a] Of course only for suitably complex problems.

Computational Complexity

Some standard complexity classes:

Name	Meaning	Examples
L	logarithmic space	graph accessibility
P	polynomial time	model checking
NP	nondeterministic pol. time	prop. logic SAT
PSpace	polynomial space	Q-SAT
ExpTime	exponential time	
NExpTime	nondeterministic exponential time	
ExpSpace	exponential space	
...	...	
	undecidable	FOL-SAT

To determine that a problem $P \subseteq M$ is

- **in** a complexity class $\mathcal{C}$, it suffices to
  - design/find an algorithm
  - show that it is sound, complete, and terminating, and
  - show that this algorithm runs, for every $m \in M$, in **at most** $\mathcal{C}$ resources
  - ...this algorithm can be a reduction to a problem known to be in $\mathcal{C}$
- **hard for** a complexity class $\mathcal{C}$, we need to
  - find a suitable problem $P' \subseteq M'$ that is **known to be hard** for $\mathcal{C}$ and
  - a reduction $\pi(.)$ from P' to P
- **complete for** a complexity class $\mathcal{C}$, we need to show that it is
  - in $\mathcal{C}$ and
  - hard for $\mathcal{C}$

Known Complexity Results so Far:

- We have seen that $\mathcal{ALC}$ concept satisfiability (no TBox) is **in PSpace**:
  - non-deterministic tableau algorithm runs in polynomial space
  - can be extended to ABoxes

✔ we can't do better: $\mathcal{ALC}$ satisfiability is **PSpace-hard**:
  - but proof is a bit cumbersome
  - via a reduction of satisfiability of quantified Boolean formulae

- We have seen that $\mathcal{ALC}$ concept satisfiability w.r.t. TBoxes is **in NExpSpace**:
  - non-deterministic tableau algorithm runs in exponential space
  - can be extended to ABoxes & ontology consistency
  - can be extended to $\mathcal{ALCQI}$, $\mathcal{ALCQO}$, and $\mathcal{ALCIO}$

✔ we **can** do better: $\mathcal{ALC}$ satisfiability wrt. TBoxes is **ExpTime-complete**:
  - but such (optimal) algorithm takes too long for this course
  - as is lower bound/hardness proof
    (via a reduction of the halting problem of polynomial-space-bounded alternating TMs)

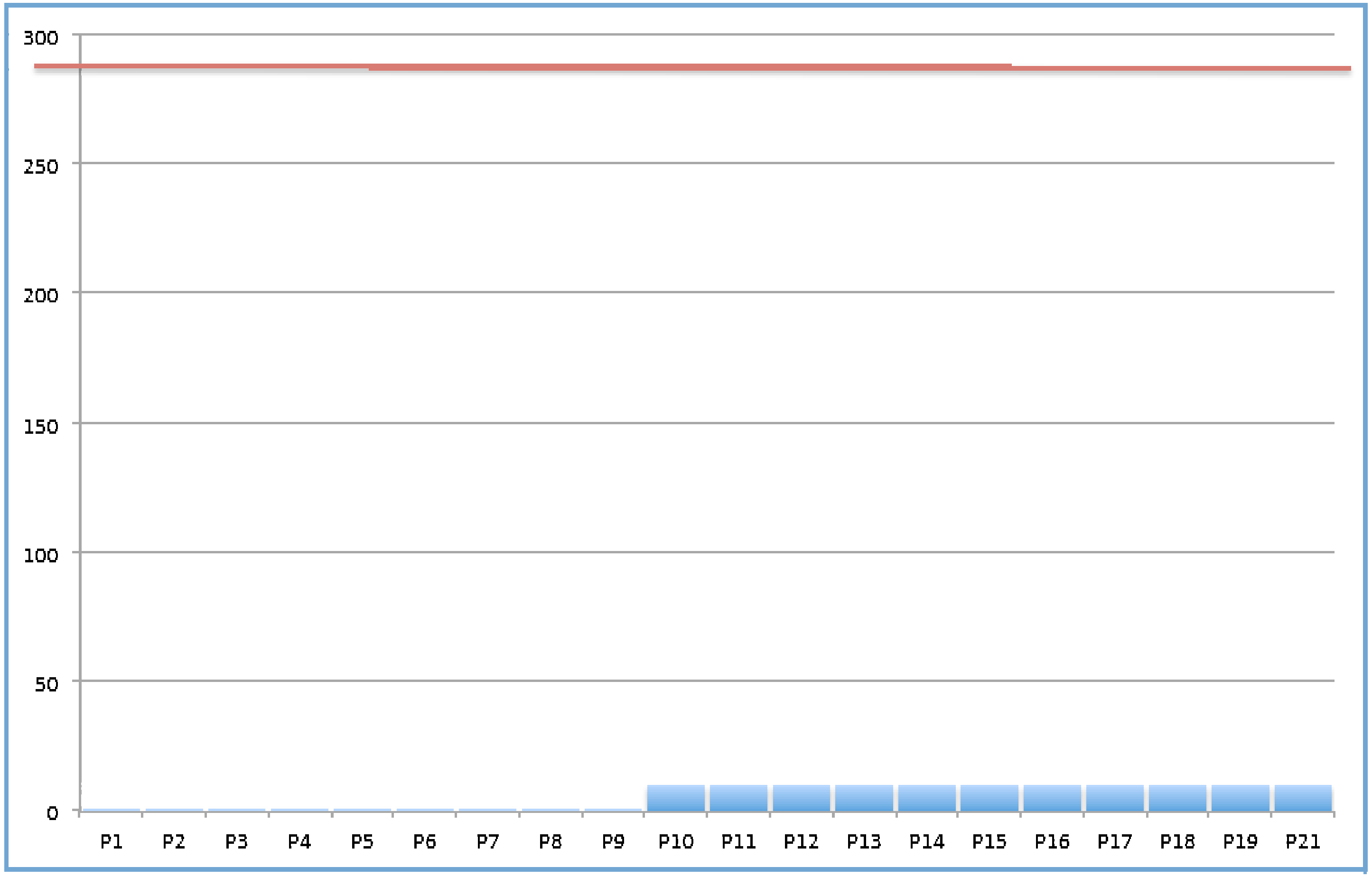
Worst-Case Complexity – why we bother

Understanding lower bounds/hardness of problems

- tell us when "in principle" improvement of algorithms is futile
- inform us of relationships of logics:
  - who is harder than who?
  - which are of similar difficulty?
- their proofs often
  - reveal interesting model theoretic properties:
    - tree model property: each satisfiable input has a tree-shaped model
    - finite model property: each satisfiable input has a finite model
  - use interesting translations between logics

They don't always tell us much about "typical" performance...

Worst-case: algorithm runs, for every $m \in M$, in **at most** $\mathcal{C}$ resources, e.g., like this, on all problems of size 7:



Worst-Case Complexity

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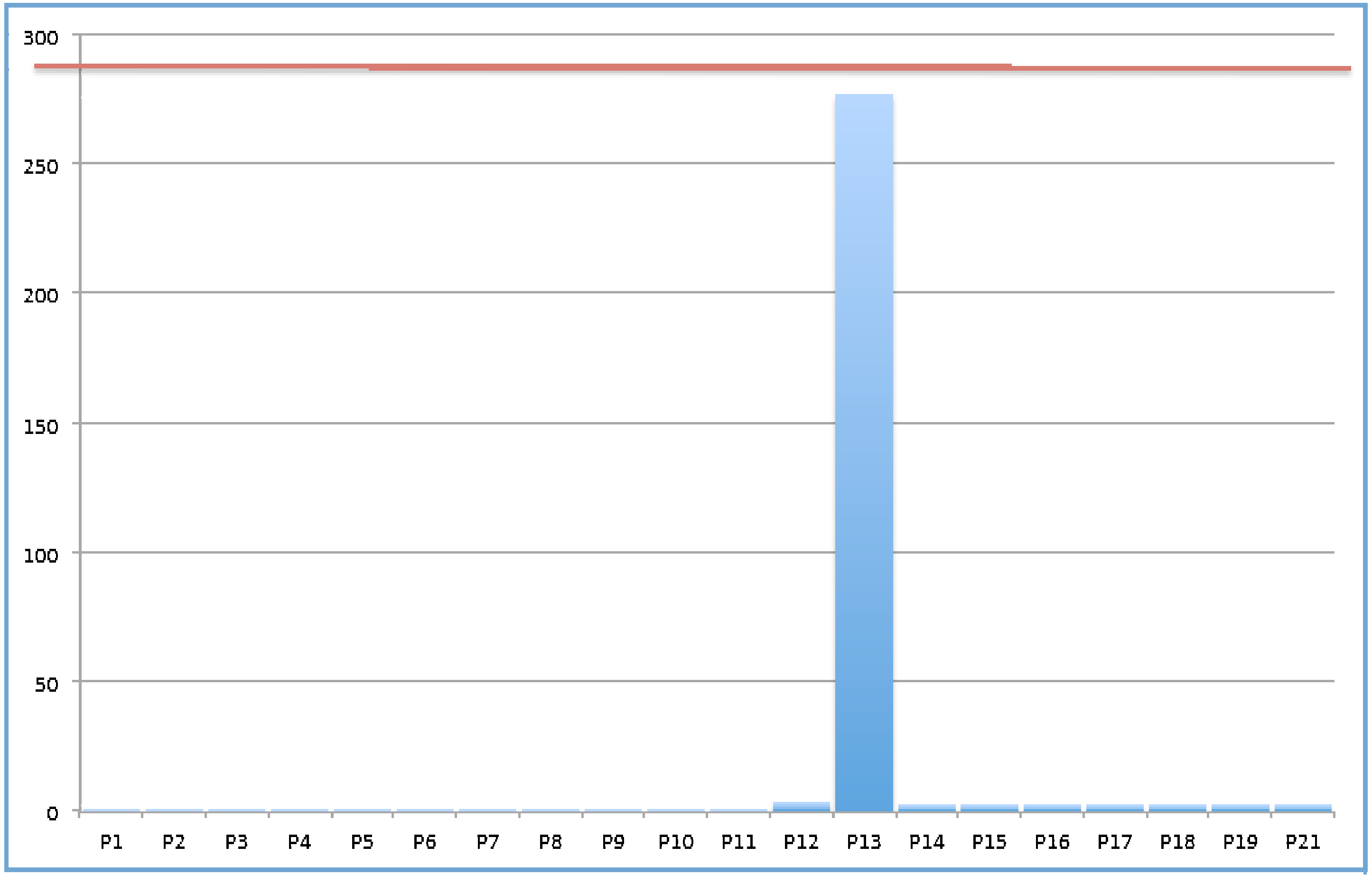

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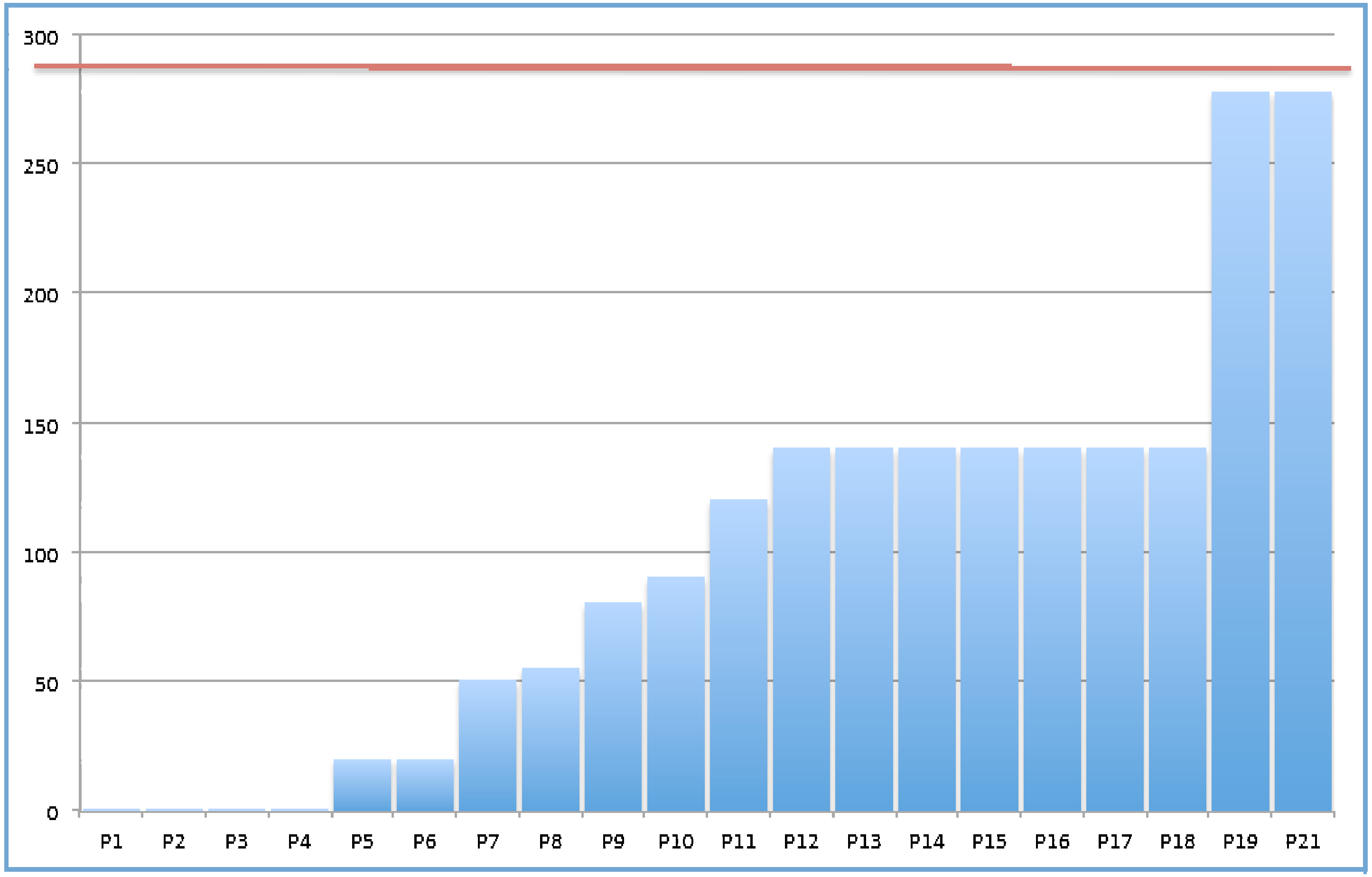
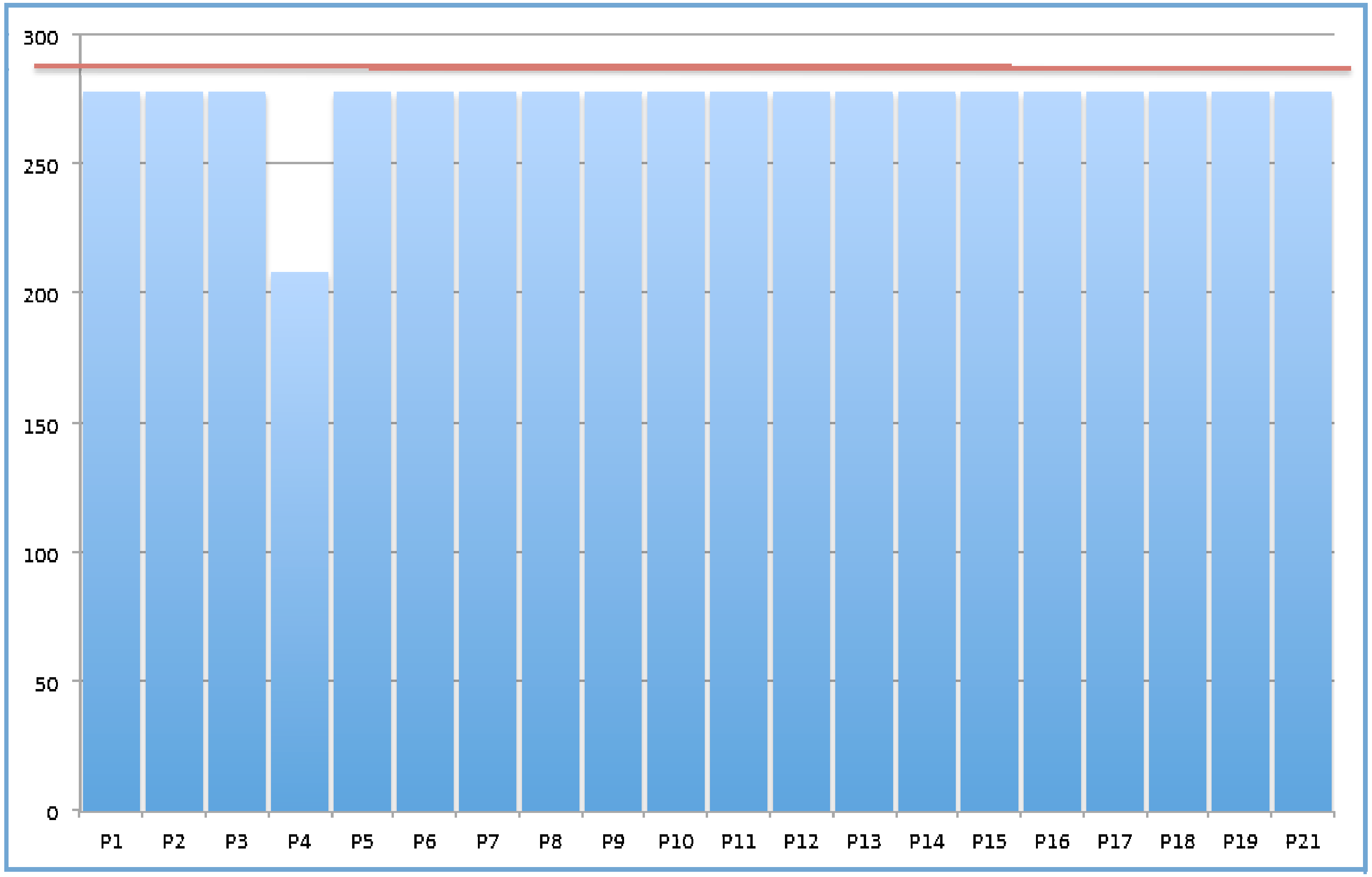


Worst-Case Complexity

Worst-case: algorithm runs, for every $m \in M$, in **at most** C resources, e.g., or like this, on all problems of size 7:



Is concept satisfiability always easier?

Earlier, we have claimed that, for $\mathcal{ALC}$,

- concept satisfiability is in PSpace, but
- concept satisfiability w.r.t. a TBox is in ExpTime

Next, we will see that, for $\mathcal{ALC}^u$, the extension of $\mathcal{ALC}$ with

- **universal role** u with $u^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$

$\Rightarrow$ concept satisfiability is as hard as reasoning w.r.t. a TBox, namely **ExpTime-hard**

- this is typical phenomenon where the
  - certain constructors enable us to **internalise** a TBox

Internalising a TBox

Remember: for $\mathcal{T} = \{C_1 \sqsubseteq D_1, \ldots, C_n \sqsubseteq D_n\}$, we use

$$C_{\mathcal{T}} = (\neg C_1 \sqcup D_1) \sqcap \ldots \sqcap (\neg C_n \sqcup D_n)$$

for the **universal $\mathcal{T}$ concept** that has to hold everywhere

Reduction: for C a concept and $\mathcal{T}$ a TBox, define

$$\pi(C, \mathcal{T}) = C \sqcap \forall u.C_{\mathcal{T}}$$

Lemma:
1. C is satisfiable w.r.t. $\mathcal{T}$ iff the concept $\pi(C, \mathcal{T})$ is satisfiable
2. the size of $\pi(C, \mathcal{T})$ is linear in that of C plus $\mathcal{T}$

Corollary: satisfiability of $\mathcal{ALC}^u$ concepts is as hard as satisfiability of $\mathcal{ALC}$ concepts w.r.t. TBoxes is, namely ExpTime-hard

Let's do that again!

Is concept satisfiability always easier? II

Earlier, we have claimed that, for $\mathcal{ALC}$,

- concept satisfiability is in PSpace, but
- concept satisfiability w.r.t. a TBox is in ExpTime

Next, we will see that, for $\mathcal{ALCIO}$, the extension of $\mathcal{ALC}$ with

- **inverse roles** r^- with $(r^-)^\mathcal{I} = \{(y,x) \mid (x,y) \in r^\mathcal{I}\}$ and
- **nominals**, i.e., individual names used as concept names

$\Rightarrow$ concept satisfiability is as hard as reasoning w.r.t. a TBox, namely **ExpTime-hard**

- this is typical phenomenon where the
  - **combination** of certain constructors enables us to **internalise** a TBox

Internalising a TBox II

Remember: for $\mathcal{T} = \{C_1 \sqsubseteq D_1, \ldots, C_n \sqsubseteq D_n\}$, we use

$$C_{\mathcal{T}} = (\neg C_1 \sqcup D_1) \sqcap \ldots \sqcap (\neg C_n \sqcup D_n)$$

for the **universal $\mathcal{T}$ concept** that has to hold everywhere

Reduction: for C a concept and $\mathcal{T}$ a TBox, define

$$\pi(C, \mathcal{T}) = C \sqcap C_{\mathcal{T}} \sqcap \exists p. \left(\{o\} \sqcap \forall p^-.(\bigsqcap_r \forall r.(\exists p.\{o\} \sqcap C_{\mathcal{T}})\right)$$

Lemma: 1. C is satisfiable w.r.t. $\mathcal{T}$ iff the concept $\pi(C, \mathcal{T})$ is satisfiable

2. the size of $\pi(C, \mathcal{T})$ is linear in that of C plus $\mathcal{T}$

Corollary: satisfiability of $\mathcal{ALCIO}$ concepts is as hard as satisfiability of $\mathcal{ALCIO}$ concepts w.r.t. TBoxes, namely ExpTime-hard

Are all DLs in ExpTime?

Earlier, we have claimed that $\mathcal{ALCQI}$, $\mathcal{ALCQO}$, and $\mathcal{ALCIO}$ are all **ExpTime**-complete, i.e., as hard/easy as $\mathcal{ALC}$

Next, we will see that consistency of $\mathcal{ALCQIO}$ ontologies, the extension of $\mathcal{ALC}$ with

- **inverse roles** r^- with $(r^-)^\mathcal{I} = \{(y,x) \mid (x,y) \in r^\mathcal{I}\}$
- **number restrictions**, in fact functionality restrictions $(\leq 1 r\ \top)$ and
- **nominals**, i.e., individual names used as concept names

$\Rightarrow$ is harder, namely **NExpTime-hard**

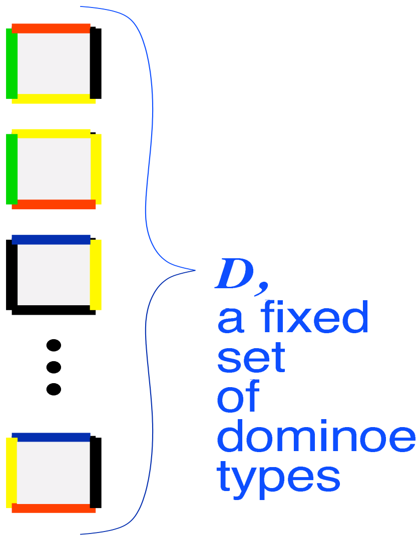
- this is typical phenomenon where
  - **combination** of otherwise harmless constructors leads to increased complexity

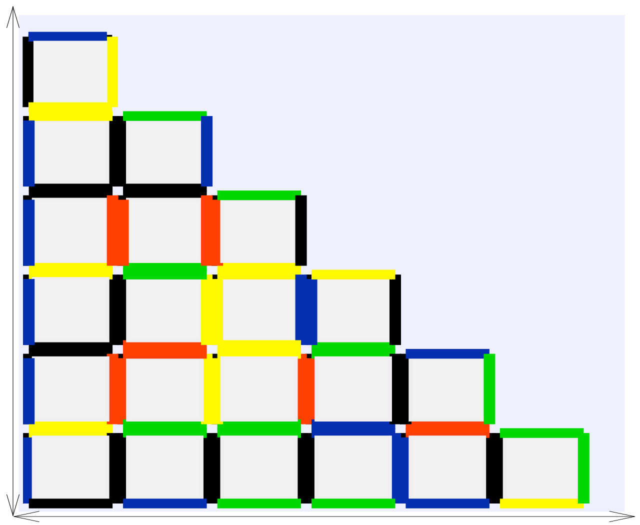
$\mathcal{ALCQIO}$ is NExpTime-hard

We follow our hardness proof recipe:

- to show that consistency of $\mathcal{ALCQIO}$ ontologies is **NExpTime-hard**, we
  - find a suitable problem $P' \subseteq M'$ that is **known to be NExpTime-hard** and
  - a reduction from P' to $\mathcal{ALCQIO}$ consistency

The **NExpTime** version of the **domino problem**





can we tile a 2^n x 2^n squar
using D?

Domino Problems

Definition: A **domino system** $\mathcal{D} = (D, H, V)$

- set of domino **types** $D = \{D_1, \ldots, D_d\}$, and
- horizontal and vertical **matching conditions** $H \subseteq D \times D$ and $V \subseteq D \times D$

A **tiling** for $\mathcal{D}$ is a function:

$$t : \mathbb{N} \times \mathbb{N} \longrightarrow D \text{ such that}$$
$$\langle t(m,n), t(m+1,n)\rangle \in H \text{ and}$$
$$\langle t(m,n), t(m,n+1)\rangle \in V$$

Domino problems: **classical** given $\mathcal{D}$, does $\mathcal{D}$ have a tiling?

$\Rightarrow$ well-known that this problem is **undecidable** [Berger66]

NexpTime given $\mathcal{D}$, does $\mathcal{D}$ have a tiling for $2^n \times 2^n$ square?

$\Rightarrow$ well-known that this problem is **NExpTime-hard**

Reduction of NExpTime Domino Problem to $\mathcal{ALCQIO}$ Consistency

To reduce the NExpTime domino problem to $\mathcal{ALCQIO}$ consistency, we need to

- define a mapping $\pi(.)$ from **domino problems** to $\mathcal{ALCQIO}$ **ontologies** such that

D has an $2^n \times 2^n$ mapping iff $\pi(D)$ is consistent
and
size of $\pi(D)$ is polynomial in n

Mapping a Domino System into an $\mathcal{ALCQIO}$ Ontology

Elements in models of $\pi(D)$ will stand for points in the grid, i.e., (m, n)...

We can express various **obligations** of the domino problem in $\mathcal{ALC}$ TBox axioms:

① each element carries **exactly one domino type** D_i

$\rightsquigarrow$ use concept name D_i for each domino type and

$$
\begin{array}{rcll}
\top & \sqsubseteq & D_1 \sqcup \ldots \sqcup D_d & \text{\% each element carries a domino type} \\
D_1 & \sqsubseteq & \neg D_2 \sqcap \ldots \sqcap \neg D_d & \text{\% but not more than one} \\
D_2 & \sqsubseteq & \neg D_3 \sqcap \ldots \sqcap \neg D_d & \text{\% ...} \\
\vdots & & \vdots & \\
D_{d-1} & \sqsubseteq & \neg D_d &
\end{array}
$$

Mapping a Domino System into an $\mathcal{ALCQIO}$ Ontology

② every element has a horizontal (X-) successor and a vertical (Y-) successor

$$\top \sqsubseteq \exists X.\top \sqcap \exists Y.\top$$

③ every element satisfies D's **horizontal/vertical matching conditions**:

$$\begin{array}{lcl}
D_1 & \sqsubseteq & \bigsqcup_{(D_1,D)\in H} \forall X.D \sqcap \bigsqcup_{(D_1,D)\in V} \forall Y.D \\
D_2 & \sqsubseteq & \bigsqcup_{(D_2,D)\in H} \forall X.D \sqcap \bigsqcup_{(D_2,D)\in V} \forall Y.D \\
\vdots & \vdots & \\
D_d & \sqsubseteq & \bigsqcup_{(D_d,D)\in H} \forall X.D \sqcap \bigsqcup_{(D_d,D)\in V} \forall Y.D
\end{array}$$

Does this suffice?
I.e., does D have a $2^n \times 2^n$ tiling iff one D_i is satisfiable w.r.t. ① to ③?

- if yes, we have shown that satisfiability of $\mathcal{ALC}$ is NExpTime-hard
- so no...what is missing?

Mapping a Domino System into an $\mathcal{ALCQIO}$ Ontology

Two things are missing:

1. the model must be large enough, namely $2^n \times 2^n$ and
2. for each element, its horizontal-vertical-successors **coincide** with their vertical-horizontal-successors and vice versa

This will be addressed using a "**counting and binding together**" trick ...

④ counting and binding together

(a) use $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ as "bits" for binary representation of **grid position**
e.g., (010, 011) is represented by an instance of $\neg A_3, A_2, \neg A_1, \neg B_3, B_2, B_1$

write GCI to ensure that X- and Y-successors are **incremented** correctly
e.g., X-successor of (010, 011) is (011, 011)
e.g., Y-successor of (010, 011) is (010, 100)

(b) use a nominal to ensure that there is only one (111. . . 1, 111. . . 1)
this implies, with $\top \sqsubseteq (\leq 1\ X^-.\top) \sqcap (\leq 1\ Y^-.\top)$ **uniqueness** of grid positions

④ counting and binding together

(a) $\tilde{A}_i$ for "**bit A_i is incremented correctly**":

$$\top \sqsubseteq \tilde{A}_1 \sqcap \ldots \sqcap \tilde{A}_n$$

$$\tilde{A}_1 \sqsubseteq (A_1 \sqcap \forall X.\neg A_1) \sqcup (\neg A_1 \sqcap \forall X.A_1)$$

$$\begin{aligned}\tilde{A}_i \sqsubseteq \quad & (\bigsqcap_{\ell<i} A_\ell \sqcap ((A_i \sqcap \forall X.\neg A_i) \sqcup (\neg A_i \sqcap \forall X.A_i)) \sqcup \\ & (\neg \bigsqcap_{\ell<i} A_\ell \sqcap ((A_i \sqcap \forall X.A_i) \sqcup (\neg A_i \sqcap \forall X.\neg A_i))\end{aligned}$$

(add the same for the B_is)

(b) ensure uniqueness of grid positions:

$A_1 \sqcap \ldots \sqcap A_n \sqcap B_1 \sqcap \ldots \sqcap B_n \sqsubseteq \{o\}$ % top right $(2^n, 2^n)$ is unique

$\top \sqsubseteq (\leq 1\ X^-.\top) \sqcap (\leq 1\ Y^-.\top)$ % everything else is also unique

Reduction of NExpTime Domino Problem to $\mathcal{ALCQIO}$ Consistency

Lemma: let $\pi(D)$ be ontology consisting of all axioms mentioned in ①-④:

- D has an $2^n \times 2^n$ tiling iff $\pi(D)$ is consistent
- size of $\pi(D)$ is polynomial (quadratic) in
  - the size of D and
  - n

Since the NExpTime-domino problem is NExpTime-hard, this implies consistency of $\mathcal{ALCQIO}$ is also NExpTime-hard:

if we **could** solve consistency of $\mathcal{ALCQIO}$ in, say, ExpTime,
this would allow us to solve the domino problem also in ExpTime via $\pi(.)$

Let’s do this again!

Are all DLs decidable?

So far, we have extended $\mathcal{ALC}$ with

- inverse role and
- number restrictions
- ...which resulted in logics whose reasoning problems are **decidable**
- ...we even discussed **decision procedures** for these extensions

Next, we will discuss some undecidable extension

- $\mathcal{ALC}$ with role chain inclusions
- $\mathcal{ALC}$ with number restrictions on complex roles

An undecidable DL: $\mathcal{ALC}$ with role chain inclusions

OWL 2 supports axioms of the form

- $r \sqsubseteq s$: a model of $\mathcal{O}$ with $r \sqsubseteq s \in \mathcal{O}$ must satisfy $r^\mathcal{I} \subseteq s^\mathcal{I}$
- **trans**(r): a model of $\mathcal{O}$ with **trans**$(r) \in \mathcal{O}$ must satisfy $r^\mathcal{I} \circ r^\mathcal{I} \subseteq r^\mathcal{I}$, where $p \circ q = \{(x, z) \mid$ there is $y : (x, y) \in p$ and $(y, z) \in q\}$, i.e., a model $\mathcal{I}$ of $\mathcal{O}$ must interpret r as a transitive relation
- $r \circ s \sqsubseteq t$: a model of $\mathcal{O}$ with $r \circ s \sqsubseteq t \in \mathcal{O}$ must satisfy $r^\mathcal{I} \circ s^\mathcal{I} \subseteq t^\mathcal{I}$

subject to some complex restrictions

...**why do we need restrictions?**

...because axioms of this form lead to **loss of tree model property** and **undecidability**

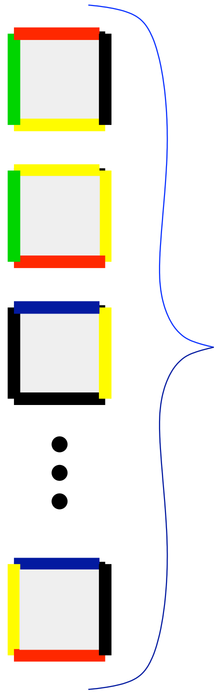
How to prove undecidability of a DL

Similar to hardness results, we prove undecidability of a DL as follows:

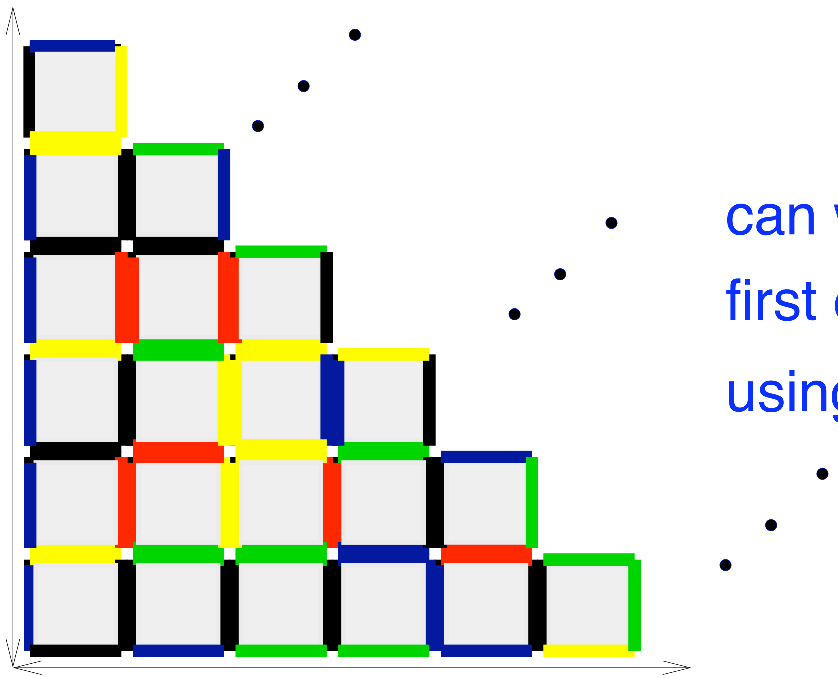
1. **fix reasoning problem**, e.g., satisfiability of a concept w.r.t. a TBox
   - remember Theorem 1?
   - if concept satisfiability w.r.t. TBox is undecidable,
   - then so is consistency of ontology
   - then so is subsumption w.r.t. TBox
   - ...
2. **pick a decision problem** known to be undecidable, e.g., **the domino problem**
3. **provide a (computable) mapping** $\pi(\cdot)$ that
   - takes an instance D of the domino problem and
   - turns it into a concept A_D and a TBox $\mathcal{T}_D$ such that
   - D has a tiling **if and only if** A_D is satisfiable w.r.t. $\mathcal{T}_D$

   i.e., a decision procedure of concept satisfiability w.r.t. TBoxes could be used as a decision procedure for the domino problem

The Classical Domino Problem



***D*,**
a fixed
set
of
dominoe
types



can we tile the
first quadrant
using ***D*?**

The Classical Domino Problem

Definition: A **domino system** $\mathcal{D} = (D, H, V)$

- set of domino **types** $D = \{D_1, \ldots, D_d\}$, and
- horizontal and vertical **matching conditions** $H \subseteq D \times D$ and $V \subseteq D \times D$

A **tiling** for $\mathcal{D}$ is a (total) function:

$$t : \mathbb{N} \times \mathbb{N} \longrightarrow D \text{ such that}$$
$$\langle t(m,n), t(m+1,n) \rangle \in H \text{ and}$$
$$\langle t(m,n), t(m,n+1) \rangle \in V$$

Domino problem: given $\mathcal{D}$, has $\mathcal{D}$ a tiling?

It is well-known that this problem is undecidable [Berger66]

Almost Encoding the Classical Domino Problem in $\mathcal{ALC}$

We have already see how to express various **obligations** of the domino problem in $\mathcal{ALC}$ TBox axioms:

① each element carries **exactly one domino type** D_i ✔

$\rightsquigarrow$ use unary predicate symbol D_i for each domino type and

$$
\begin{array}{rcll}
\top & \sqsubseteq & D_1 \sqcup \ldots \sqcup D_d & \text{\% each element carries a domino type} \\
D_1 & \sqsubseteq & \neg D_2 \sqcap \ldots \sqcap \neg D_d & \text{\% but not more than one} \\
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$$

Almost Encoding the Classical Domino Problem in $\mathcal{ALC}$

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\vdots & \vdots & \\
D_d & \sqsubseteq & \bigsqcup_{(D_d,D)\in H} \forall X.D \sqcap \bigsqcup_{(D_d,D)\in V} \forall Y.D
\end{array}$$

Does this suffice?

No, we know that it doesn't!

Encoding the Classical Domino Problem in $\mathcal{ALC}$ with role chain inclusions

④ for each element, its horizontal-vertical-successors **coincide** with their vertical-horizontal-successors & vice versa

$$X \circ Y \sqsubseteq Y \circ X \text{ and } Y \circ X \sqsubseteq X \circ Y$$

Lemma: Let $\mathcal{T}_D$ be the axioms from ① to ④.
Then $\top$ is satisfiable w.r.t. $\mathcal{T}_D$ iff $\mathcal{D}$ has a tiling.

- since the domino problem is undecidable, this implies undecidability of concept satisfiability w.r.t. TBoxes of $\mathcal{ALC}$ with role chain inclusions
- due to Theorem 1, all other standard reasoning problems are undecidable, too
- Proof: 1. show that, from a tiling for D, you can construct a model of $\mathcal{T}_D$
  2. show that, from a model $\mathcal{I}$ of $\mathcal{T}_D$, you can construct a tiling for D (tricky because elements in $\mathcal{I}$ can have several X- or Y-successors but we can simply take **the right ones**...)

Let's do this again!

Let's do this again!

What other constructors can us help to express obligation ④?

- counting and complex roles (role chains and role intersection):

$$\top \sqsubseteq (\leq 1\boldsymbol{X}.\top) \sqcap (\leq 1\boldsymbol{Y}.\top) \sqcap (\exists(\boldsymbol{X} \circ \boldsymbol{Y}) \sqcap (\boldsymbol{Y} \circ \boldsymbol{X}).\top)$$

- restricted role chain inclusions (only 1 role on RHS), and counting on **non-simple** roles:

$$\begin{aligned} \top &\sqsubseteq (\leq 1\boldsymbol{X}.\top) \sqcap (\leq 1\boldsymbol{Y}.\top) \\ \boldsymbol{X} \circ \boldsymbol{Y} &\sqsubseteq \boldsymbol{r} \\ \boldsymbol{Y} \circ \boldsymbol{X} &\sqsubseteq \boldsymbol{r} \\ \top &\sqsubseteq (\leq 1\boldsymbol{r}.\top) \end{aligned}$$

- various others...

Are all DLs hard/intractable?

Let's see a less complex DL: $\mathcal{EL}$

Thanks to **Thomas Schneider**, University of Bremen: he made the originals of these slides, which I borrowed and slightly modified

Are all DLs intractable?

1. What is $\mathcal{EL}$?
2. Normalisation
3. A simple poly-time reasoning algorithm

And now ...

1. What is $\mathcal{EL}$?
2. Normalisation
3. A simple poly-time reasoning algorithm

Summary

$\mathcal{EL}$ is a restriction of $\mathcal{ALC}$ that ...

- allows only conjunction and existential restrictions
- is at the heart of OWL 2 EL
- whose standard reasoning problems are in PTime, i.e.,

  i.e., there is a worst-case polynomial-time algorithm for deciding subsumption etc.

Summary

$\mathcal{EL}$ is a restriction of $\mathcal{ALC}$ that ...

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- can be extended to $\mathcal{EL}^{++}$ with other features, without increase in complexity:

$\perp$	domain and range restrictions
disjoint concepts	concept and role assertions
role (chain) inclusions	nominals
transitive roles, reflexive roles	concrete domains

Summary

$\mathcal{EL}$ is a restriction of $\mathcal{ALC}$ that ...

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$\perp$	domain and range restrictions
disjoint concepts	concept and role assertions
role (chain) inclusions	nominals
transitive roles, reflexive roles	concrete domains

- whose extension with inverse roles or counting increases complexity to that of $\mathcal{ALC}$

Syntax and semantics of $\mathcal{EL}$

Concepts

For C, D concepts and R a role name:

Constructor	Syntax	Example	Semantics
top	$\top$		$\Delta^{\mathcal{I}}$
conjunction	$C \sqcap D$	$\mathtt{Human} \sqcap \mathtt{Male}$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
exist. restr.	$\exists r.C$	$\exists\mathtt{hasChild.Human}$	$\{x \mid \exists y.(x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$

Axioms

- $C \sqsubseteq D$
- $C \equiv D$ as a shortcut for "$C \sqsubseteq D$, $D \sqsubseteq C$"

What a tiny logic!?

✓ We **can say** in $\mathcal{EL}$

$\mathsf{Hand} \sqsubseteq \exists\, \mathsf{hasPart}.\mathsf{Finger}$

✗ but we **can't say**

$\mathsf{Hand} \sqsubseteq {=}5\, \mathsf{hasPart}.\mathsf{Finger}$

$\mathsf{Finger} \sqsubseteq \exists\, \mathsf{hasPart}^{-}.\mathsf{Hand}$

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$\mathsf{Finger} \sqsubseteq \exists\, \mathsf{hasPart}^{-}.\mathsf{Hand}$

✗ We'd like to say, but **can't**

$\mathsf{MildFlu} \equiv \mathsf{Flu} \sqcap \forall\, \mathsf{symptom}.\neg \mathsf{Triv}$

$\mathsf{Triv} \equiv \mathsf{Cough} \sqcup \mathsf{Sneeze.} \sqcup \mathsf{Headache}$

$\mathsf{MildFlu} \equiv \mathsf{Flu} \sqcap \forall\, \mathsf{symptom}.\neg \mathsf{Fever}$

✓ all we **can say** (in $\mathcal{EL}^{++}$) is

$\mathsf{MildFlu} \sqsubseteq \mathsf{Flu}$

$\mathsf{Cough} \sqsubseteq \mathsf{Triv},\ \mathsf{Sneeze} \sqsubseteq \mathsf{Triv}, \ldots$

$\mathsf{MildFlu} \sqcap \exists\, \mathsf{symptom.Fever} \sqsubseteq \bot$

What a tiny logic!?

✓ We **can say** in $\mathcal{EL}$

$\mathsf{Hand} \sqsubseteq \exists\, \mathsf{hasPart.Finger}$

✗ but we **can't say**

$\mathsf{Hand} \sqsubseteq {=}5\, \mathsf{hasPart.Finger}$

$\mathsf{Finger} \sqsubseteq \exists\, \mathsf{hasPart}^{-}.\mathsf{Hand}$

✗ We'd like to say, but **can't**

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$\mathsf{Triv} \equiv \mathsf{Cough} \sqcup \mathsf{Sneeze}. \sqcup \mathsf{Headache}$

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✓ all we **can say** (in $\mathcal{EL}^{++}$) is

$\mathsf{MildFlu} \sqsubseteq \mathsf{Flu}$

$\mathsf{Cough} \sqsubseteq \mathsf{Triv}$, $\mathsf{Sneeze} \sqsubseteq \mathsf{Triv}, \ldots$

$\mathsf{MildFlu} \sqcap \exists\, \mathsf{symptom.Fever} \sqsubseteq \bot$

$\mathcal{EL}^{++}$ **is used** in some large-scale ontologies, e.g., SNOMED

$\mathcal{EL}^{(+)}$ is not so tiny – an example ontology

$$
\begin{array}{rcl}
\mathsf{Endocardium} & \sqsubseteq & \mathsf{Tissue} \sqcap \exists \mathsf{cont\text{-}in}.\mathsf{HeartWall} \sqcap \\
 & & \qquad \exists \mathsf{cont\text{-}in}.\mathsf{HeartValve} \\
\mathsf{HeartWall} & \sqsubseteq & \mathsf{BodyWall} \sqcap \exists \mathsf{part\text{-}of}.\mathsf{Heart} \\
\mathsf{HeartValve} & \sqsubseteq & \mathsf{BodyValve} \sqcap \exists \mathsf{part\text{-}of}.\mathsf{Heart} \\
\mathsf{Endocarditis} & \sqsubseteq & \mathsf{Inflammation} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{Endocardium} \\
\mathsf{Inflammation} & \sqsubseteq & \mathsf{Disease} \sqcap \exists \mathsf{acts\text{-}on}.\mathsf{Tissue} \\
\mathsf{Heartdisease} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{HeartValve} & \sqsubseteq & \mathsf{CriticalDisease} \\
\mathsf{Heartdisease} & \equiv & \mathsf{Disease} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{Heart}
\end{array}
$$

Taken from [Baader et al. 2006]

$\mathcal{EL}^{(+)}$ is not so tiny – an example ontology

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\mathsf{Endocarditis} & \sqsubseteq & \mathsf{Inflammation} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{Endocardium} \\
\mathsf{Inflammation} & \sqsubseteq & \mathsf{Disease} \sqcap \exists \mathsf{acts\text{-}on}.\mathsf{Tissue} \\
\mathsf{Heartdisease} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{HeartValve} & \sqsubseteq & \mathsf{CriticalDisease} \\
\mathsf{Heartdisease} & \equiv & \mathsf{Disease} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{Heart}
\end{array}
$$

$$
\mathcal{EL}^{+} \left\{
\begin{array}{rcl}
\mathsf{part\text{-}of} \circ \mathsf{part\text{-}of} & \sqsubseteq & \mathsf{part\text{-}of} \\
\mathsf{part\text{-}of} & \sqsubseteq & \mathsf{cont\text{-}in} \\
\mathsf{has\text{-}loc} \circ \mathsf{cont\text{-}in} & \sqsubseteq & \mathsf{has\text{-}loc}
\end{array}
\right.
$$

Taken from [Baader et al. 2006]

Satisfiability and subsumption

Satisfiability + coherence are trivial: every $\mathcal{EL}$-TBox is coherent

because ?

Satisfiability and subsumption

Satisfiability + coherence are trivial: every $\mathcal{EL}$-TBox is coherent

- $\mathcal{I}$ with $A^\mathcal{I} = \Delta^\mathcal{I}$ and $r^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$,
  for all concept names A and role names r,
  satisfies every $\mathcal{EL}$ axiom
- ($\mathcal{I}$ with $A^\mathcal{I} = r^\mathcal{I} = \emptyset$ doesn't – **why?**)

Satisfiability and subsumption

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Subsumption ?

Satisfiability and subsumption

Satisfiability + coherence are trivial: every $\mathcal{EL}$-TBox is coherent

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  for all concept names A and role names r,
  satisfies every $\mathcal{EL}$ axiom
- ($\mathcal{I}$ with $A^\mathcal{I} = r^\mathcal{I} = \emptyset$ doesn't – **why?**)

Subsumption isn't:
does the following TBox entail $A \sqsubseteq B$? $A' \sqsubseteq B'$?

$$\begin{aligned} \exists r.A &\sqsubseteq \exists r.B \\ A' &\equiv \exists r.\exists r.A \\ B' &\equiv \exists r.\exists r.B \end{aligned}$$

Satisfiability and subsumption

Satisfiability + coherence are trivial: every $\mathcal{EL}$-TBox is coherent

- $\mathcal{I}$ with $A^\mathcal{I} = \Delta^\mathcal{I}$ and $r^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$,
  for all concept names A and role names r,
  satisfies every $\mathcal{EL}$ axiom
- ($\mathcal{I}$ with $A^\mathcal{I} = r^\mathcal{I} = \emptyset$ doesn't – **why?**)

Subsumption isn't:
does the following TBox entail $A \sqsubseteq B$? $A' \sqsubseteq B'$?

$$\begin{aligned} \exists r.A &\sqsubseteq \exists r.B \\ A' &\equiv \exists r.\exists r.A \\ B' &\equiv \exists r.\exists r.B \end{aligned}$$

Without negation, they are not interreducible: Theorem 1 fails!

An Algorithm for $\mathcal{EL}$ subsumption

Goal: present a decision procedure for subsumption in $\mathcal{EL}$

Outline:

1. Normalisation procedure
2. Decision procedure

   (simple, naïve, without optimisations)

And now ...

1. What is $\mathcal{EL}$?
2. Normalisation
3. A simple poly-time reasoning algorithm

Normal form

...keeps the reasoning procedure simple

Definition

An $\mathcal{EL}$ ontology is in **normal form** if all axioms have these forms:

$$\begin{aligned} A_1 \sqcap \ldots \sqcap A_n &\sqsubseteq B \\ A &\sqsubseteq \exists r.B \\ \exists r.A &\sqsubseteq B \end{aligned}$$

$A_{(i)}, B$: concepts **names** r: role $n \geqslant 1$ is an integer

The normalisation procedure

- ... applies **normalisation rules** to axioms in a given TBox $\mathcal{T}$
- each rule transforms an axiom into one or several shorter ones
- old axiom is removed from $\mathcal{T}$; new axioms are added

$\leadsto$ results in an "equivalent" TBox $\mathcal{T}'$

The normalisation rules

NF1	Input	$C \equiv D$	
	Output	$C \sqsubseteq D$	$D \sqsubseteq C$
NF2	Input	$\mathbf{C} \sqsubseteq \mathbf{D}$	
	Output	$\mathbf{C} \sqsubseteq A$	$A \sqsubseteq \mathbf{D}$

$C_{(i)}$ D	arbitrary concepts
$\mathbf{C_{(i)}}$ $\mathbf{D}$	complex concepts
B	concept name
A	fresh concept name

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NF3	Input	$\exists r.\mathbf{C} \sqsubseteq D$	
	Output	$\mathbf{C} \sqsubseteq A$	$\exists r.A \sqsubseteq D$
NF4	Input	$C_1 \sqcap \ldots \sqcap \mathbf{C_i} \sqcap \ldots \sqcap C_n \sqsubseteq D$	
	Output	$\mathbf{C_i} \sqsubseteq A$	$C_1 \sqcap \ldots \sqcap A \sqcap \ldots \sqcap C_n \sqsubseteq D$

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NF4	Input	$C_1 \sqcap \ldots \sqcap \mathbf{C_i} \sqcap \ldots \sqcap C_n \sqsubseteq D$	
	Output	$\mathbf{C_i} \sqsubseteq A$	$C_1 \sqcap \ldots \sqcap A \sqcap \ldots \sqcap C_n \sqsubseteq D$
NF5	Input	$B \sqsubseteq \exists r.\mathbf{C}$	
	Output	$B \sqsubseteq \exists r.A$	$A \sqsubseteq \mathbf{C}$
NF6	Input	$B \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$	
	Output	$B \sqsubseteq C_1 \quad \ldots$	$B \sqsubseteq C_n$

The normalisation procedure

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- contains new concept names $A_1, \ldots, A_k$
- is of size linear in the size of $\mathcal{T}$
- is “equivalent” to $\mathcal{T}$ ...

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Lemma

- For every model $\mathcal{I} \models \mathcal{T}$, there is a model $\mathcal{J} \models \mathcal{T}'$ such that $X^{\mathcal{J}} = X^{\mathcal{I}}$ for all $X \notin \{A_1, \ldots, A_k\}$.
- For every model $\mathcal{J} \models \mathcal{T}'$, it holds that $\mathcal{I} \models \mathcal{T}$.

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Consequence: $\mathcal{T}'$ is equivalent to $\mathcal{T}$ w.r.t. subsumption:

$\mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T}' \models C \sqsubseteq D$

for all C, D that don't use the A_i

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Initial assumptions

Input: TBox $\mathcal{T}$, concept **names** A, B

Question: does $\mathcal{T} \models A \sqsubseteq B$ hold?

Assumption of A, B being concept *names* is no real restriction:

$$\begin{array}{rcl} \mathcal{T} & \models & C \sqsubseteq D \\ & \Updownarrow & \\ \mathcal{T} \cup \{A \equiv C,\ B \equiv D\} & \models & A \sqsubseteq B \end{array}$$

Deciding subsumptions via subsumer sets

Subsumer of A: a concept name B (or $\top$) with $\mathcal{T} \models A \sqsubseteq B$

Subsumer set $S(A)$: set that contains subsumers of A

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Representation of subsumer sets: in a labelled graph $G(\mathcal{T})$

- Nodes of $G(\mathcal{T})$ = concept names (or $\top$) in $\mathcal{T}$
- Label of node A: $S(A)$

  $B \in S(A)$ means $\mathcal{T} \models A \sqsubseteq B$
- Label of edge (A, B): set $R(A, B)$ of roles

  $r \in R(A, B)$ means $\mathcal{T} \models A \sqsubseteq \exists r.B$

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Outline of the procedure:

1. Set $S(A) = \{A, \top\}$ for every A
2. Monotonically build $G(\mathcal{T})$
   by exhaustively applying completion rules
3. Check whether $B \in S(A)$ to determine whether $\mathcal{T} \models A \sqsubseteq B$

The completion rules

R1 If $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \in \mathcal{T}$
and $A_1, \ldots, A_n \in S(X)$ but $B \notin S(X)$
then add B to $S(X)$

The completion rules

R1 If $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \in \mathcal{T}$
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R2 If $A \sqsubseteq \exists r.B \in \mathcal{T}$
and $A \in S(X)$ but $r \notin R(X, B)$
then add r to $R(X, B)$

The completion rules

R1 If $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \in \mathcal{T}$
and $A_1, \ldots, A_n \in S(X)$ but $B \notin S(X)$
then add B to $S(X)$

R2 If $A \sqsubseteq \exists r.B \in \mathcal{T}$
and $A \in S(X)$ but $r \notin R(X, B)$
then add r to $R(X, B)$

R3 If $\exists r.A \sqsubseteq B \in \mathcal{T}$
and $r \in R(X, Y)$ and $A \in S(Y)$ but $B \notin S(X)$
then add B to $S(X)$

The "naïve" subsumption algorithm [Baader et al. 2006]

Algorithm 1

Input: $\mathcal{EL}$ ontology $\mathcal{T}$
Output: $S(.)$ such that $\mathcal{T} \models A \sqsubseteq B$ iff $B \in S(A)$

$\mathcal{T}' :=$ Normalise($\mathcal{T}$) % by applying **NF1** - **NF6** exhaustively

Initialise graph for $\mathcal{T}'$:
  For each concept name A in $\mathcal{T}'$ (or $\top$)
    create a node A with $S(A) := \{A, \top\}$
    set all edge labels $R(X, Y) := \emptyset$

Exhaustively apply rules **R1-R3** to graph
Output resulting graph

Exercise

Let's apply the normalisation procedure to the TBox

$$\mathcal{T} = \{ \begin{array}{rcl} A & \sqsubseteq & B \sqcap \exists r.C\,, \\ C & \sqsubseteq & \exists s.D\,, \\ \exists r.\exists s.\top \sqcap B & \sqsubseteq & D \end{array} \}$$

and then check whether it entails

$$A \sqsubseteq D.$$

Summary

Algorithm 1 . . .

- terminates in time **polynomial** in the size of $\mathcal{T}$
- constructs a **canonical model** of $\mathcal{T}$
- is **sound** and **complete**: outputs *yes* iff $\mathcal{T} \models A \sqsubseteq B$
- is *one pass* (all subsumptions in 1 pass)
- is still slow for big ontologies:
  ...search for applicable rules over 100K concept names/nodes

Smarter versions of Algorithm 1 . . .

- are goal-oriented, “one-pass”
- are implemented in the reasoners CEL, JCEL, ... for the extension $\mathcal{EL}^{++}$
- can be extended even to the Horn fragment of $\mathcal{SHIQ}$

For details see [Baader et al. 2005, Baader et al. 2006, Kazakov 2009].

References: links

Bio-medical ontologies

- SNOMED, the systematized nomenclature of human and veterinary medicine http://en.wikipedia.org/wiki/SNOMED_CT
- GALEN http://www.opengalen.org
- GO, the Gene Ontology http://www.geneontology.org

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Masters thesis, Technische Universität Dresden, Germany, 2005.
http://lat.inf.tu-dresden.de/research/papers.html#2005

What has been left out

- Loads of complexity results
- Other complexity measures
  - data complexity, relevant for OBDA – see Misha's course on Thursday!
  - average case
- Other (reasoner) performance considerations
  - what makes reasoning hard: size, **tree-width**
  - robustness
  - robustness under (small) changes to $\mathcal{O}$ & performance homo/heterogeneity
- Other reasoning problems
  - module extraction and inseparability
  - decomposition of ontologies
  - entailment explanation and justifications

Ask us for pointers, or look at Thomas Schneider & my ESSLLI 2012 course notes

Thank you for your attention!